

# Non-linear relativistic diffusions

Z. Haba

Institute of Theoretical Physics, University of Wrocław,  
50-204 Wrocław, Plac Maxa Born 9, Poland  
email:zhab@ift.uni.wroc.pl

March 29, 2011

## Abstract

We obtain a non-linear generalization of the relativistic diffusion. We discuss diffusion equations whose non-linearity is a consequence of quantum statistics. We show that the assumptions of the relativistic invariance and an interpretation of the solution as a probability distribution substantially restrict the class of admissible non-linear diffusion equations. We consider relativistic invariant as well as covariant frame-dependent diffusion equations with a drift. In the latter case we show that there can exist stationary solutions of the diffusion equation besides the equilibrium solution corresponding to the quantum or Tsallis distributions. We define the relative entropy as a function of the diffusion probability and prove that it is monotonically decreasing in time when the diffusion tends to the equilibrium. We discuss its relation to the thermodynamic behavior of diffusing particles .

## 1 Introduction

An interaction of a particle with a many particle system is too complex to be described by a microscopic Hamiltonian theory. Some approximations for the interaction are unavoidable. If the medium is passive then an approximation of a linear master equation is sufficient [1]. In the quantum case there is a restriction on such an approximation (even if the interaction is weak) resulting from quantum statistics. The probability of a transition to a quantum state depends on the occupation number of this state. Such a dependence implies a non-linear master equation [2]. The collision terms in the Boltzmann equation [3] as well as interactions in an equation for the Wigner function [4] in quantum field theory also lead to non-linear integro-differential equations. Under the approximation of two-body interactions the non-linearities are quadratic. The approximation of binary collisions together with a non-relativistic approximation to the probability distribution of the electron gas is applied to electron-photon interactions

[5][6] leading to the Kompaneets diffusion equation for the photon distribution. The simplicity of the Kompaneets equation inspired us for a search of relativistic partial differential equations of the parabolic type.

In this paper we discuss diffusion equations from the point of view of the general principles of relativistic invariance. First, we consider relativistic invariant non-linear diffusion equations in the phase space  $(x, p)$ . In general, differential equations invariant under a group of transformations  $L$  have solutions which transform in a non-trivial way under  $L$ . We consider rotation invariant stationary solutions. They transform covariantly under the Lorentz group. After a transformation the solution depends on the velocity  $w^\mu$  of the reference frame (the boost). We may treat the frame velocity as an additional physical variable and discuss relativistic equations for densities which depend on the phase space variables  $(x, p)$  and on  $w$ . In such a case we obtain a wider class of non-linear diffusion equations similar to the Kompaneets equation.

The linear relativistic diffusions have been studied for a long time (see the reviews [7][8]). It has been shown a long time ago that Markovian relativistic diffusion in the configuration space does not exist [9][10]. The relativistic diffusion (preserving the mass-shell) of a massive particle on the phase space has been defined and studied by Schay [11] and Dudley [12]. A general class of diffusions in the phase space has been discussed in [13]. An analog of Dudley's diffusion in general relativity has been investigated in [14]. We have studied diffusions with a friction of massive and massless particles in Minkowski space [15] and on a general manifold [16]. These diffusions have an equilibrium limit determined by the friction. Linear diffusion equations are applied in the quark-gluon plasma and in heavy ion collisions [17][18][19][20] (for non-linear diffusions see [21]). The relativistic invariance in these applications is not clear because only diffusion of some momentum variables is investigated (e.g. diffusion of the rapidity). The first diffusion equation applied to relativistic phenomena appeared in the paper of Kompaneets [5] (although the electron gas was treated in a non-relativistic approximation and only the diffusion of the photon energy was considered). Non-relativistic non-linear diffusion equations appear in porous media, plasma physics, stellar dynamics and most importantly in hydrodynamics [22]. A non-linear equation in the phase space (Kramers equation) can be related to a non-linear equation in the configuration space in the high friction limit [23]. The statistical physics of non-linear diffusion equations has been discussed recently in [24][25]. It seems that power-like non-linearities in the diffusion equation are intrinsically connected with the Tsallis statistics [26][27][28][29][30].

The organization of this paper is as follows. In sec.2 we begin with relativistic quantum mechanics in order to discuss the form of the nonlinear master equation. We treat Stückelberg's proper time formalism in relativistic quantum mechanics [31][32][33] as a useful heuristic tool for a derivation of relativistic invariant equations. We consider the relativistic diffusion as a classical limit of the master equation. We propose a non-linear Lorentz covariant relativistic diffusion equation preserving the positivity and normalization of the probability

distribution. In sec.3 we recall the model of the linear relativistic diffusion (we need this simple model to fix the notation). In sec.4 we obtain a non-linear diffusive relativistic transport equation. We begin the studies of its consequences. First, we show that the transport equation has a stationary solution but no equilibrium. In sec.5 relativistic covariant drifts are discussed. It is pointed out that an additional tensor is needed in order to define the drift. A unit four-vector  $w^\mu$  which can be related to the frame velocity is introduced in order to construct the drift. The resulting diffusion equation has an equilibrium. A dependence of the physical equilibrium state on the relativistic frame of reference has been discussed in physics literature for a long time (see [34] and references quoted there). We consider quadratic and power-like non-linearities. In sec.6 we show that the linear approximation coincides with the diffusion discussed in [15] and [35](equilibrating to the Jüttner distribution, see also earlier papers cited in [7][8]). In sec.7 we define the relative entropy for the non-linear diffusions. We discuss drifts leading to the quantum equilibrium or to the Tsallis distribution. We show that the relative entropy is decreasing monotonically when the diffusion tends to its equilibrium.

## 2 Irreducible representations of the Poincare group and relativistic diffusion equations

Relativistic invariance in quantum physics should be treated by means of the representation theory of the Poincare group (irreducible representations of elementary particles are discussed). Let  $A \in SL(2, C)$  and  $\Lambda$  be a homomorphism of  $SL(2, C)$  onto  $SO(3, 1)$ . We consider one-particle states (described by a function of the momentum) transforming under an irreducible unitary representation  $U$  of the Poincare group [36][37]. When restricted to the Lorentz group the representation is expressed on eigenvectors  $|\mathbf{p}, \sigma\rangle$  of the momentum  $\mathbf{P}$  and the third component of the spin  $\Sigma_3$  ( $-2j - 1 \leq \sigma \leq 2j + 1$ )

$$U(A)|p\rangle = V(\mathcal{R}(A))|\Lambda(A^{-1})p\rangle, \quad (1)$$

where  $p^\mu p_\mu = m^2 c^2$ ,  $\mathcal{R}$  is the Wigner rotation and the matrix  $V$  is defined by  $(2j + 1)$ -dimensional representation of  $SU(2)$ . We obtain generators of the Lorentz group by differentiation (Greek indices denote space-time coordinates whereas the Latin indices refer to spatial coordinates)

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}(p), \quad (2)$$

where

$$L_{jk} = -i(p_j \frac{\partial}{\partial p^k} - p_k \frac{\partial}{\partial p^j}), \quad (3)$$

$$L_{0j} = -ip_0 \frac{\partial}{\partial p^j} \quad (4)$$

and  $\Sigma_{\mu\nu} \in su(2)$  .

We outline here the scheme which leads to a relativistic diffusion equation discussed in more detail in [38](the linear theory). For any classical observable  $O$  we can define its operator Weyl version  $\mathcal{O}$ . Then, we consider matrix elements of  $\mathcal{O}$  in the basis of the wave functions (1)

$$\mathcal{O}_{\sigma\sigma'}(p, p') = \langle p, \sigma | \mathcal{O} | p', \sigma' \rangle.$$

We define a classical function on the phase space (the Wigner matrix)  $W$  as the Fourier transform of  $\mathcal{O}$  (its limit  $\hbar \rightarrow 0$  should coincide with  $O$ )

$$W_{\sigma\sigma'}(\mathbf{x}, \mathbf{p}) = \int d\mathbf{k} d\mathbf{k}' \int \delta(\mathbf{p} - \frac{1}{2}\mathbf{k} - \frac{1}{2}\mathbf{k}') \exp(i(\mathbf{k} - \mathbf{k}')\mathbf{x}) \mathcal{O}_{\sigma\sigma'}(k, k'). \quad (5)$$

In the massless case instead of  $2j + 1$  states of spin  $j$  we have only two states of helicity  $\lambda = j$  and  $\lambda = -j$ . The formulae for generators of an irreducible representation of the Poincare group have been discussed in our earlier paper following [36][39][40][41][42].

We consider relativistic quantum mechanics formulated in an invariant way by means of the proper time  $\tau$  [31][32][33]. We look for non-linear master equations consistent with quantum mechanics. A non-linearity of the master equation arises as a consequence of the quantum statistics [2] or in a description of particles interacting by means of an electromagnetic field [43][44][45]. In the first case, the probability of a transition depends on the occupation number of the state. In the second case an electromagnetic field depends on the distribution of charged particles which is described by the density matrix  $\rho$ . If we express the electromagnetic field by its source  $\rho$  then we obtain a non-linear master equation. In general, such an equation will have a complicated nonlocal form. We consider a local approximation independent of details of particular interactions. We assume that the operators in the equation can be expressed solely by generators of the Poincare group and some scalars. We demand that after a time evolution the density matrix remains a Hermitian operator and that its trace is preserved. We additionally assume that the generator of the evolution is a quadratic function of the generators of the Poincare group (this assumption leads to the diffusion). Then, the structure of the evolution equation can be derived in a similar way as in the case of the Lindblad equation [46]. Our assumptions suggest the equation

$$\partial_\tau \rho = i[P^\mu, [P_\mu, \rho]] - \frac{\gamma^2}{8}([M^{\mu\nu}, [GM_{\mu\nu}, \rho]] + [GM^{\mu\nu}, [M_{\mu\nu}, \rho]]). \quad (6)$$

Here,  $\gamma^2$  is a diffusion constant which could be determined from the physical model of particle interactions,  $G(\rho)$  is an operator depending on  $\rho$  which is a scalar with respect to the transformations of the Lorentz group. Eq. (6) is covariant under transformations of the Lorentz group. If there is another tensor  $F_{\mu\nu}$  at our disposal then we may insert  $M_{\mu\nu} + F_{\mu\nu}$  in eq.(6). We shall discuss this possibility in sec.5.

In the linear case ( $G = 1$ ) the time evolution of an observable  $\mathcal{O}$  can be defined by the time evolution of the state  $\rho$  (here the trace defines an expectation value of the observable)

$$Tr(\rho_\tau \mathcal{O}) = Tr(\rho \mathcal{O}_\tau). \quad (7)$$

From the relation (7) between the evolution of states and the evolution of observables it follows that an evolution equation for observables also preserves the positivity of an observable and its trace. Then, from eqs.(6) and (7) we obtain (see [38] and [3][4]) a linear equation for  $W$  (defined in eq.(5)). We generalize this equation (when  $G \neq 1$ ) to a non-linear diffusion equation

$$\begin{aligned} \partial_\tau W = & p^\mu \partial_\mu^x W - \frac{1}{4} \gamma^2 (M_{\mu\nu} G M^{\mu\nu} W + W G M_{\mu\nu} M^{\mu\nu} \\ & - G M_{\mu\nu} W M^{\mu\nu} - M_{\mu\nu} W G M^{\mu\nu}) \end{aligned} \quad (8)$$

where  $G(W)$  is a Lorentz scalar but in general a non-linear function of  $W$ , the operators on the rhs of  $W$  act to the left (their position is relevant only for particles with spin, see [38]).

### 3 Linear diffusion of particles without spin

Let us write explicitly the linear version ( $G = 1$ ) of eq.(8) for particles with the spin equal zero (see [38] for a higher spin)

$$\kappa^{-2} (\partial_\tau W - p^\mu \partial_\mu^x W) = \frac{1}{2} \Delta_H^m W, \quad (9)$$

where  $\kappa^2 = \gamma^2 m^{-2} c^{-2}$  and

$$\Delta_H^m = m^2 c^2 (\partial_1^2 + \partial_2^2 + \partial_3^2) + p^j p^k \partial_j \partial_k + 3 p^k \partial_k \quad (10)$$

$k = 1, 2, 3$  and  $\partial_j = \frac{\partial}{\partial p^j}$ . The derivatives over position will have an index  $x$ , derivatives without an index are over momenta. We can take the limit  $m \rightarrow 0$  of the operator (10). Then, the limit  $m \rightarrow 0$  of eq.(9) is defined by

$$\Delta_H^0 = p^j p^k \partial_j \partial_k + 3 p^k \partial_k. \quad (11)$$

We can see that the operator (10) is the Laplace-Beltrami operator on the hyperboloid  $p^\mu p_\mu = m^2 c^2$

$$\Delta_H^m = g^{-\frac{1}{2}} \partial_j g^{jk} g^{\frac{1}{2}} \partial_k, \quad (12)$$

where

$$g^{jk} = m^2 c^2 \delta^{jk} + p^j p^k, \quad (13)$$

$g = \det(g_{jk})$  and  $g^{-1} = m^6 c^6 p_0^2$ ,  $p_0 = \sqrt{m^2 c^2 + \mathbf{p}^2}$ . Note that in terms of the spatial components of the four-vector momentum the Lorentz transformation  $\mathbf{p} \rightarrow \mathbf{p}'$  is non-linear. The invariance of the metric tensor takes the form

$$g^{jk}(\mathbf{p}') = J_{jl} g^{lr}(\mathbf{p}) J_{kr}, \quad (14)$$

where

$$J_{jl} = \frac{\partial p'^j}{\partial p^l}. \quad (15)$$

Using eq.(14) we can check by direct calculations that transforming coordinates by means of the Lorentz transformation we do not change the Laplace-Beltrami operator (what is well-known; however, the explicit calculations checking the invariance of the non-linear diffusion will be useful later on)

$$\Delta_H^m(\mathbf{p}') = \Delta_H^m(\mathbf{p}). \quad (16)$$

In the classical description we define the evolution of the probability density  $p_0\Phi$  by an adjoint equation

$$(\Phi_\tau, W) \equiv \int dx d\mathbf{p} \Phi_\tau(x, \mathbf{p}) W(x, \mathbf{p}) \equiv \int dx d\mathbf{p} \Phi(x, \mathbf{p}) W_\tau(x, \mathbf{p}) = (\Phi, W_\tau). \quad (17)$$

We define the adjoints of operators with respect to the (real) scalar product (17)

$$(\Phi, AW) = (A^*\Phi, W). \quad (18)$$

Note that  $L_{jk}^* = -L_{jk}$  and  $L_{0j}^* = i\partial_j p_0$ .

We obtain the transport equation if both sides of eq.(17) do not depend on  $\tau$ . This is the case if

$$-p^\mu \partial_\mu^x W = \frac{\kappa^2}{2} \Delta_H^m W \quad (19)$$

or equivalently

$$p^\mu \partial_\mu^x \Phi = \frac{\kappa^2}{2} \Delta_H^{m*} \Phi, \quad (20)$$

where  $\Delta_H^{m*}$  is the adjoint (18) of  $\Delta_H^m$  (this is also the adjoint in  $L^2(d\mathbf{p})$ ). We write  $\Omega = p_0\Phi$ . Then, eq.(20) can be rewritten in the form

$$p^\mu \partial_\mu \Omega = \frac{\kappa^2}{2\sqrt{g}} \partial_j (\sqrt{g} g^{jk} \partial_k \Omega). \quad (21)$$

## 4 Non-linear relativistic diffusion equations (no drift)

We restrict ourselves to particles with spin zero from now on. Eq.(8) can be expressed by spatial components  $\mathbf{p}$  of the four-vector  $p$  as

$$\partial_0 \Omega - p_0^{-1} \mathbf{p} \nabla_{\mathbf{x}} \Omega = \frac{1}{2} \kappa^2 \partial_j \left( G(\Omega) g^{jk} p_0^{-1} \partial_k \Omega \right) \quad (22)$$

or in a more elegant way

$$p^\mu \partial_\mu \Omega = \frac{\kappa^2}{2\sqrt{g}} \partial_j \left( \sqrt{g} g^{jk} \partial_k \Omega \right) \quad (23)$$

when

$$G(\Omega) = \mathcal{G}(\Omega)'. \quad (24)$$

Eq.(22) is in a divergence form

$$\partial^A J_A = 0, \quad (25)$$

where  $(x^A) = (x, \mathbf{p})$ . As a consequence the particle number  $N$  is conserved ( $x_0 = ct$ )

$$\partial_t N = \partial_t \int d\mathbf{x} d\mathbf{p} \Omega = 0. \quad (26)$$

We could show that the preservation of the probability (26) is a classical equivalent of the quantum conservation laws

$$\partial_t \text{Tr}(p_0 \mathcal{O}_t) = \partial_\tau \text{Tr}(\mathcal{O}_\tau) = 0. \quad (27)$$

We are looking for solutions of eq.(22) depending only on dimensionless energy

$$\epsilon = \beta c p_0, \quad (28)$$

where  $\beta^{-1}$  is a parameter of the dimension of the energy which we can set as equal to  $(k_B T)^{-1}$  (where  $T$  is the temperature and  $k_B$  the Boltzmann constant). Then,

$$\partial_j \Omega(\epsilon) = \beta c p_j p_0^{-1} \partial_\epsilon \Omega(\epsilon).$$

If  $\Omega$  depends only on  $\epsilon$  then eq.(22) reads

$$\partial_t \Omega_t(\epsilon) = \frac{\beta c^2 \kappa^2}{2} \left( \epsilon^{-2} \partial_\epsilon (\epsilon^3 - \epsilon m^2 c^4 \beta^2) \partial_\epsilon \mathcal{G}(\Omega_t) + m^2 c^4 \beta^2 \epsilon^{-2} \partial_\epsilon \mathcal{G}(\Omega_t) \right). \quad (29)$$

First, assume  $m > 0$ , then the stationary solution  $\partial_0 \Omega_S = 0$  is determined by the equation (note that the current (25)  $J \neq 0$ , hence  $\Omega_S$  is not an equilibrium)

$$(\epsilon^3 - \epsilon m^2 c^4 \beta^2) \partial_\epsilon \mathcal{G}(\Omega_S) + m^2 c^4 \beta^2 \mathcal{G}(\Omega_S) = R, \quad (30)$$

where  $R$  is a constant. If  $\mathcal{G}$  is known then integrating eq.(30) we obtain an implicit equation for  $\Omega_S$

$$\mathcal{G}(\Omega_S) = \frac{\epsilon}{\sqrt{\epsilon^2 - m^2 c^4 \beta^2}}. \quad (31)$$

If the stationary state  $\Omega_S(\epsilon)$  is given then eq.(31) determines  $\mathcal{G}$  as a function of  $\Omega$ .

In the massless case, let the initial  $\Omega$  depends on  $\mathbf{x}$  and  $\beta c \mathbf{p} = \mathbf{n} \epsilon$ . Then, eq.(22) reads

$$\partial_t \Omega_t(\epsilon) - c \mathbf{n} \nabla_{\mathbf{x}} \Omega_t = \frac{\beta c^2 \kappa^2}{2} \epsilon^{-2} \partial_\epsilon \epsilon^3 \partial_\epsilon \mathcal{G}(\Omega_t). \quad (32)$$

Let  $\Omega_t(\mathbf{x}, \mathbf{n}, \epsilon)$  be the solution of eq.(29) (with  $m = 0$ ,  $\mathbf{x}$  and  $\mathbf{n}$  treated as fixed parameters) then the solution of eq.(32) is  $\Omega_t(\mathbf{x} - \mathbf{n}ct, \mathbf{n}, \epsilon)$ . Solutions of eq.(30) have a discontinuity at  $m = 0$ . The limit  $m \rightarrow 0$  of eq.(31) exists but is trivial ( $\mathcal{G} = \text{const}$  corresponding to  $G = 0$ ). There exist non-trivial stationary solutions of eq.(32) (which are not a limit of the solution (31)). In fact, in the massless case the condition  $\partial_t \Omega_S = 0$  gives

$$\mathcal{G}(\Omega_S) = a^{-2} \epsilon^{-2} \quad (33)$$

where  $a$  is an arbitrary integration constant.

Power-like nonlinearities in non-relativistic diffusion equations have been discussed in [26][27]. A relativistic non-linear diffusion as a function of the rapidity has been studied in [21] and applied to the heavy ion collisions. In our model the Tsallis distribution [30][47]

$$\Omega_S = (1 + (q-1)\epsilon)^{-\frac{1}{q-1}} \quad (34)$$

results as a stationary distribution if  $m = 0$ ,  $a = q - 1$  and

$$\mathcal{G} = (q-1)^{-2} \Omega^{2q-2} (1 - \Omega^{q-1})^{-2}. \quad (35)$$

From the Lorentz invariance of the diffusion equation (22) it follows that if  $\Omega_S(\epsilon)$  is a stationary solution of eq.(22) then  $\Omega_S(\omega)$  is also a stationary solution of this equation where

$$\omega = \beta c p^\mu w_\mu \quad (36)$$

and  $w_\mu$  is an arbitrary unit time-like four-vector. We can show a more general result: assume that  $\Omega$  depends only on  $\omega$  and on  $X = x^\mu w_\mu$  then eq.(23) can be expressed in the form

$$\begin{aligned} \partial_X \Omega_X(\omega) = & \frac{\beta c^2 \kappa^2}{2} \left( \omega^{-2} \partial_\omega (\omega^3 - w^\mu w_\mu \omega m^2 c^4 \beta^2) \partial_\omega \mathcal{G}(\Omega_X) \right. \\ & \left. + w^\mu w_\mu m^2 c^4 \beta^2 \omega^{-2} \partial_\omega \mathcal{G}(\Omega_X) \right). \end{aligned} \quad (37)$$

An  $X$ -independent solution of eq.(37) can be derived in the same way as in eqs.(30)-(31) with  $\epsilon \rightarrow \omega$  and  $m^2 c^4 \rightarrow m^2 c^4 w^\mu w_\mu$ . We recover eq.(29) when  $w = (1, 0, 0, 0)$ .

## 5 Frame-dependent drifts of non-linear diffusion equations

In sec.2 we suggested that the non-linear relativistic diffusion equation is essentially unique up to an arbitrary (state dependent) diffusion strength  $G$ . In sec.4 we have shown that the relativistic invariant diffusion equation has a stationary solution  $\Omega_S$  which is not an equilibrium ( $J \neq 0$ ). In this section we construct



Lorentz covariant drifts such that the resulting diffusion equation has an equilibrium (in addition to the stationary states of sec.4). We could begin (as in sec.2) with the representation theory of the Poincare group. The first order operator  $M_{\mu\nu}$  transforms as a tensor. In order to define a drift we need another tensor to couple  $M_{\mu\nu}$  in an invariant way. As an example, an interaction with the electromagnetic field  $F^{\mu\nu}$  can be treated as a drift  $F^{\mu\nu}M_{\mu\nu}W$ . However, if the equilibrium is to be space-time independent then such a drift is excluded. We cannot couple  $x^\mu$  to  $M_{\mu\nu}$  for the same reason. The tensor  $p_\mu p_\nu$  being symmetric does not couple to  $M_{\mu\nu}$ . We need some other vectors (or tensors) to achieve such a coupling. In sec.4 we have expressed the stationary state  $\Omega_S$  as a function of the energy. We have shown that a Lorentz transformation of  $\Omega_S$  depending on the velocity  $w^\mu$  of the Lorentz frame is again a stationary solution of the diffusion equation. The velocity  $w^\mu$  is determined in the unique way. We could assume from the beginning that  $w^\mu$  is a variable in the diffusion equation (on physical grounds the equilibrium should depend on the Lorentz frame [34]). In such a case we obtain a wider class of covariant diffusion equations. We consider diffusion equations which in the rest frame and in a linear approximation reduce to the diffusion equations discussed in [11][12][15] (Boltzmann statistics). Assuming that there is an antisymmetric tensor  $A_{\mu\nu}$  available we suggest the following non-linear diffusion equation with a drift ( $M_{\mu\nu} = L_{\mu\nu}$  for a particle without spin,  $L^{*\mu\nu}$  is defined below eq.(18))

$$p^\mu \partial_\mu p_0^{-1} \Omega = \frac{1}{2} \kappa^2 i L^{*\mu\nu} G \left( i L_{\mu\nu}^* p_0^{-1} \Omega + A_{\mu\nu}(\Omega) p_0^{-1} \Omega \right). \quad (38)$$

In  $\mathbf{p}$  coordinates eq.(38) reads

$$\sqrt{g} p^\mu \partial_\mu \Omega = \frac{1}{2} \kappa^2 \partial_j \left( G g^{jk} \sqrt{g} (\partial_k \Omega + A_{k\mu} p^\mu \Omega) \right). \quad (39)$$

We are looking for equilibrium solutions  $\Omega_E$  of eq.(39) depending on  $\epsilon$ . It is easy to see that such a solution is determined by the equation (the current  $J$  of eq.(25) is zero)

$$g^{jk} p_0^{-1} \partial_k \Omega_E + A^{0j} \Omega_E = 0 \quad (40)$$

(the term  $A^{jk}$  is absent in eq.(40) because of the rotation invariance of  $\Omega_E$  and the antisymmetry of  $A^{jk}$ ).

As an example of relativistic covariant eqs.(38)-(39) we could consider an interaction of a particle with an electromagnetic field (however, this does not serve our purpose of finding new equilibria). In an electromagnetic field  $A_{\mu\nu} = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  eq.(39) takes the form

$$\partial_0 \Omega - p_0^{-1} \mathbf{p} \nabla_{\mathbf{x}} \Omega = \frac{1}{2} \kappa^2 \partial_j \left( G(\Omega) g^{jk} p_0^{-1} \partial_k \Omega + F_{k\nu} p^\nu p_0^{-1} \Omega \right). \quad (41)$$

It can be shown that if  $\Omega_S(\epsilon)$  is a stationary solution of eq.(22) then the stationary solution of eq.(41) in a scalar potential  $A_0$  is  $\Omega_S(\epsilon + A_0)$ .

We could also approach the construction of a relativistic covariant drift directly (without referring to the representation theory of the Poincare group). So, if  $\phi$  is a scalar then it follows from eq.(14) that (for an arbitrary scalar  $H(\Omega)$ ) the drift

$$H(\Omega)g^{jk}\partial_j\phi\partial_k\Omega$$

is relativistic invariant. If the equilibrium is to be independent of  $x$  then we should construct the scalar from  $p$  and another vector  $w^\mu$  ( $p^2 = m^2c^2$  is a constant). We put  $\phi = w^\mu p_\mu$  and identify  $w^\mu$  with the frame velocity. Such a choice is equivalent to setting

$$A^{\mu\nu} = \beta(w^\mu p^\nu - w^\nu p^\mu)\Omega^{-1}H(\Omega) \quad (42)$$

in eqs.(38)-(39). Choosing a special frame  $w = (1, 0, 0, 0)$  gives  $A_{0j} = \beta p_j H(\Omega)\Omega^{-1}$  and  $A_{jk} = 0$  in eq.(39). Assuming that the equilibrium  $\Omega_E$  (40) depends only on the energy (28) it is easy to see that there exists an equilibrium solution of eq.(40) if

$$\partial_\epsilon \Omega_E + H(\Omega_E) = 0. \quad (43)$$

Then,  $H(\Omega) = \Omega$  gives the Jüttner distribution [48] whereas

$$H_\sigma(\Omega) = \Omega(1 + \sigma\Omega) \quad (44)$$

leads to the quantum distributions ( $\sigma = +1$  for bosons,  $\sigma = -1$  for fermions and the Boltzmann statistics could be treated as a limit  $\sigma \rightarrow 0$ ). In general, the diffusion strength  $G$  could again be a function of  $\Omega$ .

In a rest frame, the diffusion equation (39) for scalar particles has the simple form

$$\partial_0\Omega - p_0^{-1}\mathbf{p}\nabla_{\mathbf{x}}\Omega = \frac{1}{2}\kappa^2\partial_j\left(G(\Omega)\left(g^{jk}p_0^{-1}\partial_k\Omega + \beta cp^j H(\Omega)\right)\right). \quad (45)$$

In an arbitrary frame eq.(45) can be written in a covariant divergence form (25)

$$\sqrt{g}p^\mu\partial_\mu^x\Omega = \frac{1}{2}\kappa^2\partial_j\left(GHg^{jk}\sqrt{g}\left(H^{-1}\partial_k\Omega + \exp(-\beta cp_\mu w^\mu)\partial_k\exp(\beta cp_\mu w^\mu)\right)\right). \quad (46)$$

If  $\partial_\mu^x\Omega_S = 0$  in eq.(45) and  $J \neq 0$  then  $\Omega_S$  is called a stationary state. If  $J(\Omega_E) = 0$  then  $\Omega_E$  is the equilibrium. Eq.(46) has the equilibrium solution (this covariant form of the equilibrium probability distribution is discussed in particle physics in [49], see also [34])

$$\Omega_E = \left(z\exp(\beta cw^\mu p_\mu) - \sigma\right)^{-1} \quad (47)$$

for  $H$  defined in eq.(44) (here  $z = \exp(-\mu)$  and  $\mu$  is the chemical potential; the Jüttner distribution corresponds to  $\sigma = 0$ ).

The equation with the drift  $H_q(\Omega) = \Omega^q$

$$\partial_0 \Omega - p_0^{-1} \mathbf{p} \nabla_{\mathbf{x}} \Omega = \frac{1}{2} \kappa^2 \partial_j G(\Omega) \left( g^{jk} p_0^{-1} \partial_k \Omega + \beta c p_j \Omega^q \right) \quad (48)$$

gives the Tsallis equilibrium distribution (34).

We would like to find a quantum analog of the Tsallis distribution. There are already some candidates in the literature [50]. We suggest another one resulting from the drift

$$H_{\sigma q} = \left( \Omega(1 + \sigma \Omega) \right)^q \left( (1 + \sigma \Omega)^q - \sigma \Omega^q \right)^{-1} \quad (49)$$

Eq.(43) gives an implicit equation for the equilibrium

$$\Omega_E^{1-q} - (1 + \sigma \Omega_E)^{1-q} = (q-1)(\epsilon - \mu). \quad (50)$$

Eqs.(49)-(50) have the correct limit  $q \rightarrow 1$  (quantum distribution) and  $\sigma \rightarrow 0$  (classical distribution). We show in sec.7 that the drift  $H_{\sigma q}$  allows to define a monotonic relative entropy.

If we assume that  $\Omega$  depends only on  $\epsilon$  then eq.(45) takes the simple form

$$\begin{aligned} \partial_t \Omega_t(\epsilon) = & \frac{\beta c^2 \kappa^2}{2} \epsilon^{-2} \left( \partial_\epsilon \left( (\epsilon^3 - m^2 c^4 \beta^2 \epsilon) \partial_\epsilon \mathcal{G}(\Omega_t) \right) + \right. \\ & \left. m^2 c^4 \beta^2 \partial_\epsilon \mathcal{G}(\Omega_t) + m^2 c^4 \beta^2 G H + \partial_\epsilon \left( (\epsilon^3 - m^2 c^4 \beta^2 \epsilon) G H \right) \right) \end{aligned} \quad (51)$$

The stationary distribution is determined by the equation  $\partial_t \Omega = 0$ . After an integration of the rhs of eq.(51) we obtain the first order equation for  $\Omega_S$

$$G(\Omega_S) \left( \frac{d\Omega_S}{d\epsilon} + H(\Omega_S) \right) = R(\epsilon^2 - m^2 c^4 \beta^2)^{-\frac{3}{2}} \quad (52)$$

where  $R$  is an arbitrary integration constant.  $R = 0$  gives the equilibrium (43). If  $H = \Omega$  (corresponding to the Jüttner distribution) and  $G = 1$  then eq.(52) is linear. It can be checked that the only normalizable ( $\int \Omega_S < \infty$ ) solution is at  $R = 0$ . However, there may be normalizable solutions of eq.(52) with  $R \neq 0$  if  $G \neq \text{const}$ . The obvious example is  $G \simeq H^{-1}$ .

In the limit  $m \rightarrow 0$  eq.(45) takes the form resembling the Kompaneets equation [5]

$$\partial_t \Omega - c \mathbf{n} \nabla_{\mathbf{x}} \Omega = \frac{\beta c^2 \kappa^2}{2} \epsilon^{-2} \frac{\partial}{\partial \epsilon} \left( \epsilon^3 G(\Omega) \left( \frac{\partial}{\partial \epsilon} \Omega + H(\Omega) \right) \right). \quad (53)$$

where  $\mathbf{n} = \mathbf{p}/|\mathbf{p}|^{-1}$ .

The stationary distribution is determined by

$$\frac{\partial}{\partial \epsilon} \Omega_S + H(\Omega_S) = R \epsilon^{-3} G(\Omega_S)^{-1}. \quad (54)$$

The Kompaneets equation has been derived in quantum electrodynamics of electron-photon scattering [5][6][51] with a non-relativistic approximation for the

electron velocity distribution. In such a case a factor  $\epsilon^4$  enters eq.(53) instead of  $\epsilon^3$ . In refs.[52][53]  $\epsilon^4$  is replaced by a general function  $\alpha(\epsilon)$  in order to take relativistic corrections into account. Our discussion based on eqs.(14)(39)and (46) shows that the complete differential equation in the phase space is relativistic covariant only when  $\alpha(\epsilon) = \epsilon^3$ . The Kompaneets equation has important applications in astrophysics and cosmology (Sunyaev-Zeldovitch effect). It is also studied as a mathematical model for the time dependence of the Bose-Einstein condensation [51][54][55].

## 6 Low density approximation (linear diffusion)

We recall the linear diffusion of ref.[15] and its thermodynamics [35] in order to compare it to the non-linear one. If we assume that  $\Omega$  is small then the general drift  $H$  can be expanded in a Taylor series

$$H(\Omega) = \Omega + a_2\Omega^2 + \dots$$

( $H(0) = 0$  if the solution of eq.(43) is to have the required property  $\Omega_E \rightarrow 0$  for  $\epsilon \rightarrow \infty$ ). We assume in this section that  $G(\Omega) \simeq 1$  and  $\Omega \gg \Omega^2$ . Then, eq.(45) becomes linear

$$\partial_0\Omega - p_0^{-1}\mathbf{p}\nabla_{\mathbf{x}}\Omega = \frac{1}{2}\kappa^2\partial_j\left(g^{jk}p_0^{-1}\partial_k\Omega + \beta cp_j\Omega\right). \quad (55)$$

It has the Jüttner equilibrium solution [48].

We wish to describe the time evolution in terms of some extensive (thermodynamic) functions of the probability density. It seems that the relative entropy measuring the distance between two probability distributions plays the crucial role in such a description (the relative entropy measures the speed of convergence  $\Omega_t \rightarrow \Omega_E$ , its relevance has been discovered in [56][57]; the relative entropy in nonextensive statistical mechanics is discussed in [30]) We define the Boltzmann-Kullback-Leibler relative entropy  $S_K$  of two unnormalized probability distributions  $\Omega$  and  $\Omega_E$  ( $N$  and  $N_E$  are the normalization constants)

$$S_K(\Omega; \Omega_E) = N^{-1} \int d\mathbf{x} d\mathbf{p} \Omega \ln \left( N^{-1} \Omega (\Omega_E)^{-1} N_E \right). \quad (56)$$

It is known that [58]

$$S_K(\Omega; \Omega_E) \geq 0. \quad (57)$$

An easy calculation using the diffusion equation (55) gives

$$\partial_0 S_K(\Omega; \Omega_E) = -\frac{1}{2}N^{-1} \int d\mathbf{p} d\mathbf{x} p_0^{-1} \Omega^{-1} g^{jk} (\partial_j \Omega + \beta cp_j p_0^{-1} \Omega) (\partial_k \Omega + \beta cp_k p_0^{-1} \Omega). \quad (58)$$

It follows from eqs.(57)-(58) that  $S_K(\Omega; \Omega_E)$  is a non-negative function monotonically decreasing to zero when  $t \rightarrow \infty$ .

We define the Boltzmann entropy

$$S(\Omega) = -k_B \int d\mathbf{x} d\mathbf{p} \Omega \ln \Omega \quad (59)$$

the energy

$$\mathcal{E} = c \int d\mathbf{p} d\mathbf{x} p_0 \Omega \quad (60)$$

and the free energy

$$\mathcal{F} = \beta^{-1} N \left( S_K(\Omega; \Omega_E) - \ln(N N_E^{-1}) \right). \quad (61)$$

Then, from eqs.(58)-(61) we obtain the basic thermodynamic relation

$$TS = \mathcal{E} - \mathcal{F}. \quad (62)$$

## 7 Thermodynamics of the non-linear diffusion

In general, a meaningful (non-Hamiltonian) evolution equation should lead to an increase of entropy. Conversely, if the entropy is defined then the evolution equation could be determined from the requirements that the entropy is increasing and bounded. If we define the entropy  $S$  in terms of an entropy density  $s$

$$S = - \int d\mathbf{x} d\mathbf{p} s(\Omega) \quad (63)$$

then for the diffusion (22) (without drift)

$$\partial_0 S = \frac{1}{2} \int d\mathbf{x} d\mathbf{p} p_0^{-1} s''(\Omega) G(\Omega) \partial_j \Omega g^{jk} \partial_k \Omega. \quad (64)$$

It follows that  $S$  is increasing if

$$s'' \geq 0. \quad (65)$$

The diffusion without a drift has no equilibrium but only a stationary state. There may be no limit for the increase of the entropy. For diffusions with the drift (44) quadratic in  $\Omega$  (quantum equilibrium distributions) as well as for  $H_q = \Omega^q$  and  $H_{\sigma q}$  (49) (Tsallis distributions) we can define the relative entropy  $S_K(\Omega, \Omega_E)$  which is monotonically decreasing to the equilibrium state when  $t \rightarrow \infty$  ( $S_K$  of eq.(58) would not be a monotonic function). For the drift (44) define

$$S_K^\sigma(\Omega, \Omega_E) = \int d\mathbf{x} d\mathbf{p} \left( \Omega \ln(\Omega \Omega_E^{-1}) - \sigma(1 + \sigma\Omega) \ln \left( (1 + \sigma\Omega)(1 + \sigma\Omega_E)^{-1} \right) \right) \quad (66)$$

for  $H = \Omega^q$  (such a relative entropy is suggested also in the Appendix of [59]; for a general theory see [30])

$$S_K^q(\Omega, \Omega_E) = (2-q)^{-1}(1-q)^{-1} \int d\mathbf{x}d\mathbf{p} \Omega^{2-q} - (1-q)^{-1} \int d\mathbf{x}d\mathbf{p} \Omega \Omega_E^{1-q} + (2-q)^{-1} \int d\mathbf{x}d\mathbf{p} \Omega_E^{2-q}. \quad (67)$$

We define Tsallis relative quantum entropy for the drift  $H_{\sigma q}$  (49)

$$\begin{aligned} S_K^{\sigma q}(\Omega, \Omega_E) &= (2-q)^{-1}(1-q)^{-1} \int d\mathbf{x}d\mathbf{p} \Omega^{2-q} \\ &- (1-q)^{-1} \int d\mathbf{x}d\mathbf{p} \Omega \Omega_E^{1-q} + (2-q)^{-1} \int d\mathbf{x}d\mathbf{p} \Omega_E^{2-q} \\ &- \sigma(2-q)^{-1}(1-q)^{-1} \int d\mathbf{x}d\mathbf{p} (1+\sigma\Omega)^{2-q} + \sigma(1-q)^{-1} \int d\mathbf{x}d\mathbf{p} (1+\sigma\Omega)(1+\sigma\Omega_E)^{1-q} \\ &- \sigma(2-q)^{-1} \int d\mathbf{x}d\mathbf{p} (1+\sigma\Omega_E)^{2-q}. \end{aligned} \quad (68)$$

The three entropies (66)-(68) have the properties  $S_K(\Omega_E, \Omega_E) = 0$ ,  $S'_K(\Omega_E, \Omega_E) = 0$  and  $S''_K(\Omega, \Omega_E) > 0$ . It follows that  $S_K(\Omega, \Omega_E) > 0$  (in contradistinction to  $S_K$  in eq.(58) we do not need to normalize  $\Omega$  and  $\Omega_E$  in order to prove the positivity of  $S_K$  in (66)-(68)).

In an equilibrium state

$$J^r = G(\Omega_E) g^{rk} p_0^{-1} \left( \partial_k \Omega_E + \beta c p_k p_0^{-1} H(\Omega_E) \right) = 0. \quad (69)$$

Then, using eq.(39) for the calculation of the time derivative of  $S_K$  we obtain

$$\partial_0 S_K(\Omega, \Omega_E) = -\frac{1}{2} \int d\mathbf{x}d\mathbf{p} p_0^{-1} H(\Omega)^{-1} G(\Omega) g^{jk} (\partial_j \Omega + A_{j\mu} p^\mu \Omega) (\partial_k \Omega + A_{k\mu} p^\mu \Omega) \quad (70)$$

(the Lorentz invariance of eq.(70) follows from eq.(14)). In the rest frame

$$\begin{aligned} \partial_0 S_K(\Omega, \Omega_E) &= -\frac{1}{2} \int d\mathbf{x}d\mathbf{p} p_0^{-1} H(\Omega)^{-1} G(\Omega) \\ &g^{jk} (\partial_j \Omega + \beta c p_j p_0^{-1} H(\Omega)) (\partial_k \Omega + \beta c p_k p_0^{-1} H(\Omega)) \end{aligned} \quad (71)$$

where  $H$  is either  $H_\sigma, H_q$  or  $H_{\sigma q}$ .

In the massless case eq.(71) simplifies to

$$\partial_0 S_K(\Omega, \Omega_E) = -\frac{1}{2} \int d\mathbf{p} p_0^{-1} H(\Omega)^{-1} G(\Omega) (p^j \partial_j \Omega + \beta c p_0 H(\Omega))^2. \quad (72)$$

It follows that  $S_K(\Omega, \Omega_E) > 0$  and the time evolution will decrease  $S_K$  until it achieves its minimum at the equilibrium (then  $S_K(\Omega_E, \Omega_E) = 0$ ). If the state depends solely on the energy then (this equality has been derived in the particular case of the Kompaneets equation in ref.[53])

$$\partial_0 S_K(\Omega, \Omega_E) = -4\pi \int d\epsilon \epsilon^3 G H^{-1} (\partial_\epsilon \Omega + H)^2. \quad (73)$$

In the stationary state  $\Omega_S$  (54) (when  $J \neq 0$ ) there is a correction to the formulas (70)-(73). So, in the massless case under the assumption that both  $\Omega$  and  $\Omega_S$  depend only on  $\epsilon$  we obtain

$$\begin{aligned} \partial_0 S_K(\Omega, \Omega_S) &= -4\pi \int d\epsilon \epsilon^3 G(\Omega) H(\Omega)^{-1} (\partial_\epsilon \Omega + H(\Omega))^2 \\ &+ 4\pi R \int d\epsilon G(\Omega) G(\Omega_S)^{-1} H(\Omega_S)^{-1} (\partial_\epsilon \Omega + H(\Omega)). \end{aligned} \quad (74)$$

The  $R$ -correction has an indefinite sign. Eq.(74) shows that  $S_K(\Omega, \Omega_S)$  is not monotonic.  $\Omega_t$  does not tend to  $\Omega_S$ . It is convergent to  $\Omega_E$ .

We define the entropy (as suggested by eqs.(66)-(68)) for the Bose-Einstein and Fermi-Dirac distributions

$$\mathcal{S}_\sigma = -k_B \int d\mathbf{x} d\mathbf{p} \left( \Omega \ln \Omega - \sigma(1 + \sigma\Omega) \ln(1 + \sigma\Omega) \right) \quad (75)$$

and for the quantum Tsallis distribution

$$\begin{aligned} \mathcal{S}_{\sigma q} = & -k_B(2-q)^{-1}(1-q)^{-1} \int d\mathbf{x} d\mathbf{p} \Omega^{2-q} \\ & + k_B \sigma(2-q)^{-1}(1-q)^{-1} \int d\mathbf{x} d\mathbf{p} (1 + \sigma\Omega)^{2-q} \end{aligned} \quad (76)$$

(when  $\sigma = 0$  we have  $\mathcal{S}_{\sigma q} \rightarrow \mathcal{S}_q$ ). In the models discussed in this section the principle of the maximum of entropy is satisfied.  $S$  is achieving the maximum (with fixed  $\int \Omega = 1$  and  $\int \epsilon \Omega = \mathcal{E}$ ) at the equilibrium state  $\Omega_E$ . We define the free energy as  $\mathcal{F} = \mathcal{E} - T\mathcal{S}$  (where  $\mathcal{E}$  is defined in eq.(60)). From eqs.(66)-(68) and (75)-(76) we obtain that  $\mathcal{F} = S_K + \mathcal{F}_E$  where  $\mathcal{F}_E$  is the time independent equilibrium free energy. Then, eq.(62) is satisfied as an identity.

## 8 Discussion and summary

The non-linear diffusion equations result from a macroscopic averaging of complex microscopic phenomena. We have shown that the basic requirements imposed on such equations determine their form. The linear part of the equation is already determined by the Lorentz invariance and the requirement of the positivity of the probability density. The non-linearity of the equation comes from quantum statistics (dependence of the transition rate to a state on the occupation of the state) and from an elimination of some fields which are averaged in the reduced dynamics. We have shown that the form of the non-linearity is closely related with the formula for the entropy. It is possible to obtain a drift for a linear diffusion such that the diffusion is equilibrating to the Bose-Einstein or Fermi-Dirac equilibrium distribution [15]. However, it seems that the non-linear drift is indispensable if we wish to define the relative entropy as a monotonically decreasing function which satisfies the thermodynamic relation (62). We can apply the non-linear diffusion equation to a statistical description of the motion of a stream of relativistic particles in a medium of some other relativistic particles. Such a problem is studied in high-energy physics [3] and astrophysics [60][61]. However, the whole space-time dependence of the phase-space distribution is not discussed in these papers. It can be a complicated problem to derive an equation of space-time evolution of the stream of particles on the basis of a theory of fundamental interactions (see [6][62][63][64]). It may require non-perturbative methods. In general, the resulting equation will be non-local. If the Markovian approximation is applicable to the full space-time dynamics then the relativistic invariance leads to the equations discussed in this

paper. The result of the multi-particle ultra-relativistic scattering may be independent of the details of the interaction but depend more on kinematics and the relativistic processes of dissipation. In such a case we could check in experiments assumptions underlying the dissipative dynamics of eq.(6). Diffusion equations already served as a theoretical basis for RHIC data analysis [20][21]. The diffusion model could be tested in heavy ion collision experiments. The complete space-time diffusion equation is also interesting as a model for space-time evolution of the Bose-Einstein condensation (discussed in atomic physics [54] and in astrophysics as a model for the star formation from the dark matter [65]).

## References

- [1] E.M. Lifshits and L.P. Pitaevskij,  
Physical Kinetics,Pergamon Press,1981
- [2] G. Kaniadakis,Physica, **A296**,405(2001)  
G. Kaniadakis and P. Quarati,  
Physica **A237**,229(1997)  
Phys.Rev.**E49**,51039(1994)  
F.D. Nobre, E.M.F. Curado and G. Rowlands,  
Physica **A334**,109(2004)
- [3] S.R. de Groot, W.A. van Leeuwen and Ch.G. van Weert, Relativistic Kinetic Theory, North Holland,1980
- [4] P. Carruthers and F. Zachariasen,  
Phys.Rev.**D13**,950(1976)
- [5] A.S. Kompaneets, Sov.Phys.-JETP,**4**,730(1957)
- [6] G.B. Rybicki and A.P. Lightman,  
Radiative Processes in Astrophysics,Wiley-VCH,1979
- [7] C. Chevalier and F. Debbasch,  
AIP Conf.Proc.**913**,42(2007)
- [8] J. Dunkel and P. Hänggi,Phys.Rep.**471**,1(2009)
- [9] J. Lopuszanski, Acta Phys.Pol.**12**,87(1953)
- [10] R. Hakim, Journ.Math.Phys.**9**,1805(1968)
- [11] G.Schay,PhD thesis,Princeton University,1961



- [12] R.Dudley, Arkiv for Matematik,**6**,241(1965)
- [13] C. Chevalier and F. Debbasch,  
Journ.Math.Phys.**49**,043303(2008)
- [14] J. Franchi and Y. Le Jan,  
Comm.Pure Appl.Math.**60**,187(2007)
- [15] Z. Haba, Phys.Rev.**E79**,021128(2009)
- [16] Z. Haba, Class.Quant.Grav.**27**,095021(2010)
- [17] R.C. Hwa,Phys.Rev.**D32**,637(1985)
- [18] B.Svetitsky, Phys.Rev.**D37**,2484(1988)
- [19] D.B. Walton and J. Rafelski,Phys.Rev.Lett.**84**,31(2000)
- [20] R.Rapp and H. van Hees, arXiv:0903.1096
- [21] G. Wolschin,Phys. Rev.**C69**,024906(2004),  
Europhys. Lett.**74**,29(2006), Ann. Physik,**17**,462(2008);  
W. M. Alberico, P. Czerski, A. Lavagno, M. Nardi and V. Soma, Physica,  
**A387**,467(2008)
- [22] Ya.B. Zeldovich and Yu.P. Raizer, Physics of Shock Waves and High Temperature Hydrodynamic Phenomena, Academic Press,1967
- [23] P.H. Chavanis,Phys.Rev.**E68**,036108(2003)
- [24] L. Borland, F. Pennini, A.R. Plastino and A. Plastino, Eur.Phys.J.  
**B12**,289(1999)
- [25] T.D. Frank,Phys.Lett.**A305**,150(2002)
- [26] C. Tsallis and D.J. Bukman, Phys.Rev.**E54**,R2197(1996)
- [27] A.R. Plastino and A. Plastino, Physica **A222**,347(1995)
- [28] C. Tsallis in [29]
- [29] "Nonextensive Statistical Mechanics and Its Applications" S.Abe and Y. Okamoto, Eds.,Springer LNP 560,Berlin, 2001
- [30] C. Tsallis, Introduction to Nonextensive Statistical Mechanics:Approaching a Complex World, Springer, New York,2009
- [31] E.C.G. Stückelberg,  
Helv.Phys.Acta,**14**,372(1941),**14**,588(1941)

- [32] R.P. Feynman, Phys.Rev.**80**,440(1950),**84**,108(1951)
- [33] L.P. Horwitz,S. Shashoua and W.C. Schieve, Physica **A161**,300(1989)
- [34] R.K. Pathria, Proc.Phys.Soc.**88**,791(1966)  
J.H. Eberly and A. Kujawski, Phys.Rev.**155**,109(1967)  
P.J.E. Peebles and D.T. Wilkinson,  
Phys.Rev.**174**,2168(1968)
- [35] Z.Haba,Mod.Phys.Lett.**A25**,2683(2010)
- [36] S. Weinberg, The Quantum Theory of Fields, Vol.1, Cambridge,1995
- [37] Y. Ohnuki, Unitary Representations of the Poincare Group and Relativistic Wave Equations, World Scientific,1988
- [38] Z.Haba, Journ.Phys.**A42**,445401(2009)
- [39] H.E. Moses, Journ.Math.Phys.**9**,2039(1968)
- [40] L.L. Foldy, Phys.Rev.**122**,275(1961)
- [41] Yu.M.Shirokov, JETP, **33**,1208(1957) (in Russian)
- [42] I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys.Rev. **D35**,1383(1987)
- [43] G.Lenz, P.Meystre and E.M. Wright,  
Phys.Rev.Lett.**71**,3271(1993)
- [44] Y.Castin and K. Molmer,Phys.Rev.**A51**,R3426(1995)
- [45] Y. Castin and K. Molmer, Phys.Rev.**A54**,5275(1996)
- [46] G. Lindblad, Commun.Math.Phys.**48**,119(1976)  
V. Gorini, A. Kossakowski and E.C.G. Sudarshan,  
Journ.Math.Phys.**17**,821(1976)
- [47] C.Tsallis, Journ.Stat.Phys.**52**,479(1988)
- [48] F. Jüttner, Ann.Phys.(Leipzig)**34**,856(1911)
- [49] E. Mac and E. Ugaz, Zeit.Phys.**C43**,655(1989)  
T. Matsui, B. Svetitsky and L.D. McLerran,  
Phys.Rev.**D34**,783(1986)

- [50] U. Tirnakli, F. Büyükkilic, D. Demirhan,  
Phys.Lett. **A245**,62(1998)  
C. Tsallis in [29]  
H.G. Miller,F.C. Khanna, R. Teshima, A.R. Plastino and A. Plastino,  
Phys.Lett. **A359**,357(2006)
- [51] Ya.B. Zeldovich,Sov.Phys.-Usp.**18**,79(1975)
- [52] G. Cooper, Phys.Rev.**D3**,2312(1971)  
G. Chaplin, G. Cooper and S. Slutz,  
Phys. Rev.**D9**,1273(1974)
- [53] R.E. Caflisch and C.D.Levermore,  
Phys.Fluids, **29**,748(1986)
- [54] Ch. Josserand, Y.Pomeau and S. Rica,  
Journ.Low Temp.Phys.**145**,231(2006)
- [55] M. Escobedo, M.A. Herrero and J.J.L. Velazquez,  
Physica **D126**,236(1999)and TAMS **350**,3837(1998)
- [56] J.L. Lebowitz and P.G. Bergmann,  
Ann. Phys.(N.Y.)**1**,1(1957)
- [57] H. Haken and R. Graham,  
Zeitsch.Phys.**243**,289(1971),**245**,141(1971)
- [58] H. Risken, The Fokker-Planck Equation, Springer,1989
- [59] A.K. Rajagopal in [29]
- [60] J. Bernstein and S. Dodelson, Phys.Rev.**D41**,354(1990)
- [61] W. Hu, D. Scott and J. Silk, Phys.Rev.**D49**,648(1994)
- [62] H.-Th. Elze and U. Heinz, Phys.Rep.**183**,81(1989)
- [63] M. Askawa, S.A. Bass and B. Müller,  
Progr.Theor.Phys.**116**,725(2007)
- [64] Z.Haba and H. Kleinert, Eur.Phys.J.**B21**,553(2001)
- [65] D.V. Semikoz and I.I. Tkachev,  
Phys.Rev.Lett.**74**,3093(1995)